Characteristic roots and Characteristic vector

Cayley-Hamilton Theorem & its Applications

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Cayley-Hamilton Theorem

In linear algebra, the Cayley-Hamilton theorem (named after the mathematicians Arthur Cayley and William Rowan Hamilton) states that every square matrix (with real or complex entries) satisfies its own characteristic equation.

If $\boldsymbol{A}$ is a given $n \times n$ matrix and I_n is the $n \times n$ identity matrix, then the characteristic polynomial of $\boldsymbol{A}$ is defined as

$$p(t) = |\mathbf{A} - tI_n|,$$

where t is a variable for a scalar element of the base ring. The Cayley-Hamilton theorem states that if one defines an analogous matrix equation, $p(\boldsymbol{A})$, consisting of the replacement of t with the matrix $\boldsymbol{A}$, then this polynomial in the matrix $\boldsymbol{A}$ results in the zero matrix,

$$p(\boldsymbol{A}) = \boldsymbol{0}.$$

Theorem

If $p(t)$ is the characteristic polynomial for an $n \times n$ matrix $\boldsymbol{A}$, then the matrix $p(\boldsymbol{A})$ satisfies $p(\boldsymbol{A}) = \boldsymbol{0}$.

Proof. Let A be a square matrix with characteristic polynomial $p(t) = |A - tI_n| = c_0t^n + c_1t^{n-1} + c_2t^{n-2} + \cdots + c_n$. Then, we have to show that $c_0A^n + c_1A^{n-1} + c_2A^{n-2} + \cdots + c_nI_n = O$.

First, observe that $|At - I_n| = t^n|A - t^{-1}I_n| = c_0 + c_1t + c_2t^2 + \cdots + c_nt^n$. Now Laplaces formula for calculating the determinant gives the standard equation

$$|I_n - tA|I_n = (I_n - tA)adj(I_n - tA)$$

where adj(M) denotes the adjugate (or classical adjoint) of matrix M. If we consider formal power series in t, then (I_ntA) is invertible and $(I_n - tA)^{-1} = \sum_{i=0}^{\infty} A^it^i$. So

$$\left(\sum_{i=0}^{\infty} A^it^i\right)\left(c_0 + c_1t + c_2t^2 + \cdots + c_nt^n\right)I_n = adj(I_n - tA).$$

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Writing $adj(I_n - tA)$ as a formal power series in t we have $adj(I_n - tA) = \sum_{i=0}^{\infty} B_i t^i$. Therefore from last identity we have

$$\left(\sum_{i=0}^{\infty} A^i t^i\right)\left(c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n\right) I_n = \sum_{i=0}^{\infty} B_i t^i.$$

Observe that the entries in $adj(I_n - tA)$ are polynomials in t of degree at most $n-1$. So B_i is the zero matrix for $i \geq n$. Equating the coefficients of t^n on both sides gives

$c_0 A^n + c_1 A^{n-1} + c_2 A^{n-2} + \cdots + c_n I_n = O.$

Example.

Let A=$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$. The characteristic polynomial $p(t)$ of A is

$$\begin{aligned} p(t) = |A - tI_2| &= \begin{bmatrix} 1-t & 1 \\ 1 & 3-t \end{bmatrix} \\ &= t^2 - 4t + 2. \end{aligned}$$

Then the Cayley-Hamilton theorem says that the matrix $p(A) = A^2 - 4A + 2I_2$ is the 2×2 zero matrix. One can directly check this:

$$\begin{aligned} p(A) = A^2 - 4A + 2I &= \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix} + \begin{bmatrix} -4 & -4 \\ -4 & -12 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Now we discuss some problems in matrix algebra which can be solved using the Cayley-Hamilton theorem. Therefore, the following problems can be treated as applications of Cayley-Hamilton theorem

Problem 1 (Calculation of matrix polynomial)

Let T=$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Calculate and simplify the expression of matrix polynomial $T^3 + 4T^2 + 5T - 2I_3$, where I_3 is the 3×3 identity matrix.

Solution. To obtain the characteristic polynomial for T, we note that the matrix T is upper triangular. Thus $T - tI_3$ is also upper triangular and recall that the determinant of an upper triangular matrix is the product of the diagonal entries. Thus the characteristic polynomial $p_T(t)$ for T is

$$p_T(t) = \det(T - tI_3) = (1-t)(1-t)(2-t) = -t^3 + 4t^2 - 5t + 2.$$

By the Cayley-Hamilton theorem, we have $p_T(T) = -T^3 + 4T^2 - 5T + 2I_3 = O$. Here O is the 3×3 zero matrix. Now we compute

$$\begin{aligned}
-T^3 + 4T^2 + 5T - 2I &= (-T^3 + 4T^2 - 5T + 2I) + (10T - 4I) \\
&= p_T(T) + 10T - 4I = 10T - 4I \\
&= \begin{bmatrix} 10 & 0 & 20 \\ 0 & 10 & 10 \\ 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 0 & 20 \\ 0 & 6 & 10 \\ 0 & 0 & 16 \end{bmatrix}.
\end{aligned}$$

Hence the answer.

Problem 2 (Computation of inverse of a matrix)

Find the inverse matrix of the matrix A= $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ using the CayleyHamilton theorem.

Solution. To apply the Cayley-Hamilton theorem, we first determine the characteristic polynomial $p_A(t)$ of the matrix A. Let I_3 be the 33 identity matrix. Therefore we have

$$\begin{aligned}
p_A(t) &= \|A - tI| \\
&= \begin{vmatrix} 7-t & 2 & -2 \\ -6 & -1-t & 2 \\ 6 & 2 & -1-t \end{vmatrix} \\
&= (7-t) \begin{vmatrix} -1-t & 2 \\ 2 & -1-t \end{vmatrix} - 2 \begin{vmatrix} -6 & 2 \\ 6 & -1-t \end{vmatrix} + (-2) \begin{vmatrix} -6 & -1-t \\ 6 & 2 \end{vmatrix} \\
&\text{(by the first row cofactor expansion)} \\
&= -t^3 + 5t^2 - 7t + 3.
\end{aligned}$$

Therefore the Cayley-Hamilton theorem yields that $p_A(A) = -A^3 + 5A^2 - 7A + 3I_3 = O$, where O is the 3×3 zero matrix. Rearranging terms, we have

$$\begin{aligned}
&A^3 - 5A^2 + 7A = 3I_3 \\
&\Leftrightarrow A(A^2 - 5A + 7I_3) = 3I_3 \\
&\Leftrightarrow A\left(\frac{1}{3}(A^2 - 5A + 7I_3)\right) = I_3 \\
&\Leftrightarrow A^{-1} = \frac{1}{3}(A^2 - 5A + 7I_3).
\end{aligned}$$

Therefore, we have $A^{-1} = \frac{1}{3} \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$.

Problem 2 (Expression of inverse matrix from eigen values)

A matrix A is a 3×3 matrix with eigenvalues $= i, i, -1$. Check whether the A is invertible? If so, find an expression for A^{-1} as a linear combination of positive powers of A.

Solution. The determinant of a matrix is the product of its eigenvalues. So, $|A| = i.(i).(-1) = -1$. Because the determinant is non-zero, the matrix A is non-singular, and thus is invertible. Next, To find an expression for A^{-1}, we will use the Cayley-Hamilton theorem. First we find the characteristic polynomial of A, which is $p(\lambda) = (\lambda - i)(\lambda + i)(\lambda + 1) = \lambda^3 + \lambda^2 + \lambda + 1$. Therefore, Cayley-Hamilton theorem yields $A^3 + A^2 + A + I = \mathbf{0}$. Rewriting this, we have $I = -AA^2A^3 = A(-IAA^2)$. Multiplying on the left by A^{-1} yields the desired equation, $A^{-1} = -IAA^2$.

Exercises

1. Find the inverse matrix of the matrix A= $\begin{bmatrix} 1 & 1 & 2 \\ 9 & 2 & 0 \\ 5 & 0 & 3 \end{bmatrix}$ using the Cayley-Hamilton theorem.

2. Let A, B be 2×2 matrices satisfying the relation $A = AB - BA$. Prove that $A^2 = O$, where O is the 2×2 zero matrix.

3. Let A and B be 2×2 matrices such that $(AB)^2 = O$, where O is the 2×2 zero matrix. Determine whether $(BA)^2$ must be O as well. If so, prove it. If not, give a counter example.

4. A matrix A is a 3×3 matrix with eigenvalues $= i, i, 0$. Check whether A is invertible? If so, find an expression for A^{-1} as a linear combination of positive powers of A. If A is not invertible, explain why.